# Introduction to Vector Calculus and New Operators

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While the operators we will be seeing won't seem to be too important for now, understanding them will provide an analytical insight to the deeply conceptual concepts we will be discussing. Numbers are there to simplify thoughts and learning how to maneuver through the numbers using concepts such as vector calculus is not only helps simplify our understanding, but might also lead us into discovering some new properties of the functions we are studying.

## Gradient

For a real-valued function $f(x, y, z)$ on $\mathbb{R}^3$, the gradient $\nabla f(x, y, z)$ is a vector-valued function on $\mathbb{R}^3$, that is, its value at a point $(x, y, z)$ is the vector

$$\nabla f(x, y, z) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{1}$$

in $\mathbb{R}^3$, where each of the partial derivatives is evaluated at the point $(x, y, z)$. So in this way, you can think of the symbol $\nabla$ as being "applied" to a real-valued function $f$ to produce a vector $\nabla f$.

It turns out that the divergence and curl can also be expressed in terms of the symbol $\nabla$. This is done by thinking of $\nabla$ as a vector in $\mathbb{R}^3$, namely

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{2}$$

## Divergence

The divergence for a vector field$\mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$ is the dot product of $\mathbf{f}$ with $\nabla$ (thought of as a vector):

$$\begin{aligned}
\nabla \cdot \mathbf{f} &= \left( \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot (f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}) \\
&= \left( \frac{\partial}{\partial x} \right)(f_1) + \left( \frac{\partial}{\partial y} \right)(f_2) + \left( \frac{\partial}{\partial z} \right)(f_3) \\
&= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\
&= \text{div } \mathbf{f}
\end{aligned}$$

# Curl

The curl for a vector field $\mathbf{f}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + Z(x, y, z)\mathbf{k}$ is the cross product of $\mathbf{f}$ with $\nabla$ (thought of as a vector):

$$\begin{aligned}
\nabla \times \mathbf{f} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix} \\
&= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} - \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \\
&= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \\
&= \text{curl } \mathbf{f}
\end{aligned}$$

# Laplacian

For a real-valued function $f(x, y, z)$, the Laplacian of $f$, denoted by $\Delta$f, is given by

$$\nabla f(x, y, z) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{-3}$$

$$\begin{aligned}
\text{div } \nabla f &= \nabla \cdot \nabla f \\
&= \left( \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot \left( \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \right) \\
&= \frac{\partial}{\partial x}\left( \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y}\left( \frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z}\left( \frac{\partial f}{\partial z} \right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$